

The Dade group of Mackey functors for p -groups

Olcaý Coşkun¹

Boğaziçi Üniversitesi, Matematik Bölümü, Bebek, İstanbul, Turkey

and

*Feza Gürsey Center for Physics and Mathematics, Boğazici University, Kandilli 34684
İstanbul, Turkey*

Abstract

We introduce endo-permutation Mackey functors and the Mackey-Dade group of a p -group and study their basic properties. We exhibit relations between the Mackey-Dade group of a finite p -group P and the Dade groups of the Weyl groups of all subgroups of P . As a result, we show that the Mackey-Dade group tensored with \mathbb{Q} over \mathbb{Z} is the kernel of the linearization map for Mackey functors, introduced by the author.

Keywords: endo-permutation modules, Dade group, Mackey functor, ring of subquotients.

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1. Introduction

In [11], Dade introduced the family of endo-permutation kP -modules as an efficient way of studying representations of p -groups over a field of characteristic p . As shown by Dade, the class of endo-permutation modules behaves nicely under the usual operations of taking duals, taking direct summands, restriction and inflation. He also constructed a deflation map on endo-permutation modules, now called Dade's slash construction. A framework to study endo-permutation modules is the Dade group.

The structure of the Dade group, and hence the classification of all endo-permutation modules, is determined by Bouc in [6], where results of Alperin, Bouc, Bouc-Thévenaz, Bouc-Mazza and Carlson-Thévenaz are used. We refer to [7] and references therein for a complete account of the classification.

After the classification problem is solved, there appear two kinds of generalizations of endo-permutation modules. In [13] and in [20], there appear two attempts to generalize the notion of an endo-permutation module to an arbitrary finite group G . Whereas in [14], the Dade group of a fusion system over a finite p -group is constructed.

Email address: olcay.coskun@boun.edu.tr (Olcaý Coşkun)

In this paper, we introduce another of generalization of endo-permutation kP -modules by defining endo-permutation Mackey functors. We show that it is possible to construct a group structure on the class of endo-permutation Mackey functors for P , called the Mackey-Dade group of P . It turns out that the Mackey-Dade group shares some of the interesting properties of the Dade group. For example, we show that the torsion-free rank of the Mackey-Dade group of P is the number of conjugacy classes of non-cyclic subquotients, whereas, by a result of Bouc and Thévenaz, [5, Theorem A], that of the Dade group is the number of conjugacy classes of non-cyclic subgroups of P . Note that the result for the Dade group also follows from [5, Theorem D], where it is shown that the Dade group tensored with \mathbb{Q} over \mathbb{Z} is the kernel of the well-known linearization map from the Burnside ring to the ring of rational characters. We obtain the result on the rank of the Mackey-Dade group by constructing an isomorphism between this group and the product of all Dade groups of Weyl groups of all subgroups of P . Existence of such an isomorphism also suggests a relation between the Mackey-Dade group, the ring of subquotients of P , and the linearization map for Mackey functors, both introduced by the author in [10]. We explain this relation in Section 6.

It is shown in [5] that, after extending the coefficients to a field of characteristic zero, the Dade group functor becomes a p -biset functor. Unfortunately, we do not have a version of this result for the Mackey-Dade group. The difficulty is with the tensor induction map.

The organization of the paper is as follows. In Section 2, we collect necessary preliminaries on Mackey functors. Section 3 introduces a new duality functor on the category of Mackey functors which is used to determine inverse elements in the Mackey-Dade group of a finite p -group. The definition and basic properties of endo-permutation Mackey functors are discussed in Section 4 where we also introduce the Mackey-Dade group. The last two sections relate the Mackey-Dade group to known constructions. In Section 5, we show that the Mackey-Dade group of P is a product of Dade groups of the groups $N_P(Q)/Q$ as Q runs over conjugacy classes of subgroups of P and in Section 6, we show that, after extending the coefficients to a field of characteristic zero, the Mackey-Dade group is isomorphic to the kernel of the linearization map introduced in [10].

2. Preliminaries

In this section, we collect some results on Mackey functors that will be used throughout the paper. We refer to [1], [4] and [18] for details and proofs. Throughout the paper, we assume that k is a field and all modules are finitely generated left modules over the given algebra, unless otherwise specified, and G denotes a finite group. Note that although certain constructions and results are valid also over a commutative ring, our main results are only valid over fields. Thus we work over a field to simplify statements.

2.1. Mackey functors

Let G be a finite group and k be a field. Consider the free k -algebra on variables c_H^g, r_K^H, t_K^H where $K \leq H \leq G$ and $g \in G$. We define the *Mackey algebra* $\mu_k(G)$ for G over k as the quotient of this free algebra by the ideal generated by the following six relations:

- (1) $c_H^h = r_H^H = t_H^H$ for any $H \leq G$ and any $h \in H$.
- (2) $c_{g'H}^g c_H^g = c_H^{g'g}$ and $r_L^K r_K^H = r_L^H$ and $t_K^H t_L^K = t_L^H$ for any $L \leq K \leq H \leq G$ and $g, g' \in G$.
- (3) $c_K^g r_K^H = r_{gK}^H c_H^g$ and $c_H^g t_K^H = t_{gK}^H c_K^g$ for any $g \in G$ and $K \leq H \leq G$.
- (4) $r_J^H t_K^H = \sum_{x \in J \setminus H/K} t_{J \cap^x K}^J c_{J^x \cap K}^x r_{J^x \cap K}^K$ for $J \leq H \leq K$ (Mackey Relation)
- (5) $\sum_{H \leq G} r_H^H = 1$
- (6) All other products of generators are zero.

It is known that, letting H and K run over the subgroups of G and letting g run over the double coset representatives $HgK \subset G$ and letting L run over representatives of the subgroups of $H^g \cap K$ up to conjugacy, the elements $t_{gL}^H c_L^g r_L^K$ run (without repetitions) over the elements of a k -basis for the Mackey algebra $\mu_k(G)$ (cf.[19, Section 3]).

We denote by $\rho_k(G)$, called the *restriction algebra* for G over k , the subalgebra of the Mackey algebra generated by c_H^g and r_K^H for $K \leq H \leq G$ and $g \in G$. We denote by $\tau_k(G)$ the *transfer algebra* for G over k , defined as the subalgebra generated by c_H^g and t_K^H for $K \leq H \leq G$ and $g \in G$. The *conjugation algebra*, denoted $\gamma_k(G)$, is the subalgebra generated by the elements c_H^g . When there is no ambiguity, we write $\mu = \mu_k(G)$, and $\rho = \rho_k(G)$ and $\tau = \tau_k(G)$ and $\gamma = \gamma_k(G)$. The restriction algebra ρ has generators $c_J^g r_J^K$, the transfer algebra τ has generators $t_{gJ}^K c_J^g$ and the conjugation algebra γ has generators c_J^g .

We refer to [1], [9], [12], [18] and [19] for further theoretical background on Mackey functors. In this paper, we use any of the following three definitions of a Mackey functor for G over k : A Mackey functor M for G over k is

- (i) a functor from the category of subgroups of G to the category $k\text{-mod}$ of finite-dimensional k -vector spaces,
- (ii) a functor from the category of finite G -sets to $k\text{-mod}$,
- (iii) a finitely generated left $\mu_k(G)$ -module.

In [18], Thévenaz and Webb constructed an anti-automorphism $(-)^*$ of the Mackey algebra $\mu_k(G)$ which allow us to define a right $\mu_k(G)$ -module structure on Mackey functors. With this right module structure, we form, in the category $\text{Mack}(G)$ of Mackey functors, the tensor product functor $-\hat{\otimes}_{\mu_k(G)}-$ and the hom functor $\text{Hom}_{\text{Mack}(G)}(-, -)$. These functors satisfies the usual properties of the tensor product and the hom functor for module categories. We refer to [1, Chapter 1] for further details.

For our purposes, we need multiplicative functors between categories of Mackey functors preserving certain properties (e.g. being endo-permutation). It turns out that the usual functors of induction, inflation, and deflation defined in

[18] are not suitable. A general theory of functors between categories of Mackey functors is described by Bouc in [1] in details, together with their left and right adjoints. Fortunately, two of the multiplicative functors that we are looking for are among these functors. Below, we include the definition of the multiplicative inflation and the multiplicative deflation from [1] together with the properties of these functors that are used in this paper. Note that these are not naturally equivalent to the usual inflation and deflation functors introduced in [18].

Following [18], for $H \leq G$ and $N \trianglelefteq G$, we write Res_H^G and $\text{Inf}_{G/N}^G$ for the ordinary restriction and inflation functors, respectively. We also have the following two functors. For $N \trianglelefteq G$, define the multiplicative deflation functor (cf. [1, Example 9.9.2])

$$\text{Jef}_{G/N}^G : \text{Mack}(G) \rightarrow \text{Mack}(G/N)$$

so that for a Mackey functor M for G and $N \leq K \leq G$, we have

$$\text{Jef}_{G/N}^G(M)(K/N) = M(K) \Big/ \left(\sum_L \text{Im}(t_L^K : M(L) \rightarrow M(K)) \right)$$

where L runs over all subgroups of K not containing N . The transfer, restriction and conjugation maps are the ones induced by those of M on the corresponding quotients.

In the reverse direction, we denote by

$$\text{Jnf}_{G/N}^G : \text{Mack}(G/N) \rightarrow \text{Mack}(G)$$

the functor $\iota_{G/N}^G$ described in [1, Section 9.9.3]. In general, it is difficult to determine the evaluations of the Mackey functors $\text{Jnf}_{G/N}^G M$. However, by the remark at the end of Section 9.9 in [1], we have

$$\text{Jnf}_{G/N}^G M(1) = M(N/N). \quad (1)$$

Now let G and G' be groups and $\lambda : G \rightarrow G'$ be an isomorphism. Then we denote by $\text{Iso}_{G,G'}^\lambda$ the functor that transports Mackey functors for G' to Mackey functors for G via the isomorphism λ .

A final operation on Mackey functors that is used throughout the paper is the bar construction. Let M be a Mackey functor for G over a field k and $H \leq G$. Following Thévenaz and Webb [18, Section 2], we define the $kN_G(H)/H$ -module $\overline{M}(H)$ by

$$\overline{M}(H) = M(H) \Big/ \left(\sum_{L < H} \text{Im } t_L^H : M(L) \rightarrow M(H) \right).$$

Here the action of $N_G(H)/H$ is given by the conjugation maps $c_{N_G(H)/H}^g$ for $g \in N_G(H)/H$, (cf. [18, Proposition 3.4]). Via this definition, we associate a sequence $\overline{M} = (\overline{M}(H))_{H \leq G}$ of modules to any Mackey functor M . It is clear that the sequence \overline{M} is a $\gamma_k(G)$ -module, and that $\overline{M}(1) = M(1)$ by definition.

On the other hand, by the remark at the end of Section 10.8 in [1], we have

$$\overline{M}(H) = (\text{Jef}_{N_G(H)/H}^{N_G(H)} \text{Res}_{N_G(H)}^G M)(H/H). \quad (2)$$

The following proposition describes the compositions of the above functors with the bar construction. Note that the proof follows from the above isomorphism and the properties of Bouc's general construction \mathcal{L} given in [1].

Proposition 1. (Bouc) *Let G be a finite group, $N \trianglelefteq G$ and $H \leq G$. Let M be a Mackey functor for G over k and M' be a Mackey functor for G/N over k .*

1. *Given $H \leq K \leq G$, there is an isomorphism*

$$\overline{\text{Res}_K^G M}(H) \cong \text{Res}_{N_K(H)/H}^{N_G(H)/H}(\overline{M}(H))$$

of $kN_K(H)/H$ -modules.

2. *Suppose $N \leq H \leq G$. Then there is an isomorphism of $kN_G(H)/H$ -modules*

$$\overline{\text{Jef}_{G/N}^G M}(H/N) \cong \overline{M}(H)$$

where we identify the groups $N_G(H)/H$ and $(N_G(H)/N)/(H/N)$ via the canonical quotient map.

3. *There is an isomorphism*

$$\overline{\text{Jnf}_{G/N}^G M'}(H) \cong \text{Inf}_{N_G(H)/N_{HN}(H)}^{N_G(H)/H}(\text{Iso}^\lambda(\overline{M'}(HN/N)))$$

of $kN_G(H)/H$ -modules where Iso^λ denotes the transport of structure map induced by the canonical isomorphism

$$\lambda : \frac{(N_G(H)N)/N}{(HN)/N} \rightarrow \frac{N_G(H)/H}{(N_G(H) \cap N)H/H}.$$

3. Twin-dual construction

In this section, we introduce the twin-dual of a Mackey functor M on a finite group G , as a variation of the duality $M \mapsto \text{Hom}_k(M, k)$.

First recall the twin functor construction of Thévenaz [17]. Given a Mackey functor M for G , the *twin functor* TM of M is the Mackey functor such that for any subgroup $Q \leq G$, we have

$$TM(Q) = \left(\bigoplus_{K \leq Q} \overline{M}(K) \right)^Q$$

where $(?)^Q$ denotes the submodule consisting of the Q -fixed points under the conjugation action of Q coming from the Mackey functor structure. See Section 9 of [19] for the action of the Mackey algebra. In this paper, we slightly modify this construction to define the twin-dual functor of a Mackey functor. Given a γ -module C , write C^* for its dual defined by $C^*(Q) = (C(Q))^* = \text{Hom}_k(C(Q), k)$ for any subgroup Q of G . Then we define the *twin-dual* M° of a Mackey functor M to be the Mackey functor mapping $Q \leq G$ to the k -module

$$M^\circ(Q) = \left(\bigoplus_{K \leq Q} (\overline{M}(K))^* \right)_Q$$

where $(X)_Q$ denotes the Q -coinvariants of X , i.e. the largest quotient of X on which Q acts trivially. The action of the Mackey algebra is induced by its action on the Mackey functor M . Furthermore, by Theorem 3.4.1 and Proposition 3.6 in [9], we have

$$\overline{M^\circ}(Q) = (\overline{M}(Q))^* \text{ for all } 1 \leq Q \leq G. \quad (3)$$

The following result clarifies the relation between the twin-dual and the dual of a Mackey functor. The proof of the result follows easily from the duality theorems in [9, Section 4] and by [17, Theorem 12.2]. Recall that, by our standing assumption, any Mackey functor is a finite-dimensional k -vector space.

Proposition 2. *Let M be a Mackey functor for G over the field k . Then there is an isomorphism*

$$M^\circ \cong (TM)^*$$

of Mackey functors. In particular, if k has characteristic not dividing the order of G , the twin-dual M° is isomorphic to the dual M^ of M .*

4. Endo-permutation Mackey functors

For the rest of the paper, let k be an algebraically closed field of characteristic $p > 0$ and P be a finite p -group. In this section, we introduce endo-permutation Mackey functors for finite p -groups and weakly-permutation Mackey functors for any finite group.

Definition 3. *A Mackey functor M for G is called a **weakly-permutation Mackey functor** if for any subgroup H of G , the $kN_G(H)/H$ -module $\overline{M}(H)$ is a permutation module. In particular, $\overline{M}(1) = M(1)$ is a permutation kG -module.*

In [4, Definition 3.2], Bouc defines a permutation Mackey functor to be a Mackey functor isomorphic to B_X for some G -set X . One can show that any permutation Mackey functor is a weakly-permutation Mackey functor, for example by [2, Lemma 5.10]. However, in general, there are weakly-permutation Mackey functors which are not of the form B_X . Indeed, any permutation Mackey functor is projective and hence it is sufficient to exhibit a non-projective weakly-permutation Mackey functor. As an example, consider the Mackey functor

$$M := \bigoplus_{H \leq_G G} S_{H,1}^\mu$$

where the sum is over conjugacy classes of subgroups of G and $S_{H,1}^\mu$ denotes the simple Mackey functor with minimal group H and $S_{H,1}^\mu(H) \cong k$. It can be shown that for any subgroup $H \leq G$, we have $\overline{M}(H) \cong k$ as $kN_G(H)/H$ -modules, but M is not projective unless the characteristic of k does not divide the order $|N_G(H)/H|$ for each H in G . Therefore, when k has characteristic p ,

and G has order divisible by p , the functor M is an example of a non-projective weakly-permutation Mackey functor which is not a permutation Mackey functor.

We are now ready to introduce the main concept of this paper.

Definition 4. *Let P be a finite p -group and k be a field of characteristic p . A Mackey functor M for P is called an **endo-permutation Mackey functor** if the Mackey functor $M^\circ \widehat{\otimes} M$ for P is a weakly-permutation Mackey functor.*

Note that, by [1, Proposition 1.9.1], there is an isomorphism of Mackey functors $M^\circ \widehat{\otimes} M \cong M^\circ \widehat{\otimes} M$. Thus, equivalent to the above definition, a Mackey functor M is an endo-permutation Mackey functor if $M^\circ \widehat{\otimes} M$ is a weakly-permutation Mackey functor.

It is possible to characterize endo-permutation Mackey functors in terms of endo-permutation kP -modules, introduced by Dade in [11]. Given a finite p -group P and a field k of characteristic p , a kP -module M is called an *endo-permutation kP -module* if $\text{End}_k(M)$ is a permutation kP -module. Here $\text{End}_k(M)$ is a kP -module via the action given by

$$(g \cdot \phi)(m) = g \cdot \phi(g^{-1} \cdot m)$$

for all $g \in G$, $\phi \in \text{End}_k(M)$ and $m \in M$. Note that, putting $M^* = \text{Hom}_k(M, k)$, we have $\text{End}_k(M) = \text{Hom}_k(M, M) \cong M^* \otimes M$.

Lemma 5. *Let M be a Mackey functor for P over k . Then the following statements are equivalent.*

1. *M is an endo-permutation Mackey functor.*
2. *For each subgroup Q of P , the $kN_P(Q)/Q$ -module $\overline{M}(Q)$ is an endo-permutation module.*

Proof By [1, Lemma 10.8.5], we have

$$\overline{M^\circ \widehat{\otimes} M}(Q) \cong \overline{M}(Q) \otimes \overline{M^\circ}(Q)$$

where the tensor product on the right hand side is the usual tensor product over k of $kN_P(Q)/Q$ -modules. Also by the isomorphism given in (3), we have $\overline{M^\circ}(Q) = (\overline{M}(Q))^*$. Therefore, we get

$$\overline{M^\circ \widehat{\otimes} M}(Q) \cong (\overline{M}(Q))^* \otimes \overline{M}(Q).$$

Now the result follows from the definition of an endo-permutation module. \square

As remarked in [18, p. 1874], the Mackey algebra is a finite dimensional k -algebra and hence Krull-Schmidt Theorem applies and we have projective modules, simple modules, indecomposable modules etc. Also, by [16] (cf. [18, Section 11]), there is a theory of vertices and sources for projective and simple Mackey functors. Now these remarks and the above result allow us to define a capped endo-permutation Mackey functor.

Definition 6. An endo-permutation Mackey functor M for P is called **capped** if for each subgroup $Q \leq P$, the endo-permutation $kN_P(Q)/Q$ -module $\overline{M}(Q)$ is capped.

Before looking at the properties of endo-permutation Mackey functors, we include some examples.

Example 1. Any weakly-permutation Mackey functor M for P over k is also an endo-permutation Mackey functor. Indeed, if M is weakly-permutation, then the twin-dual functor M° is also weakly-permutation by Equation (3), since the dual of a permutation module is also a permutation module. Hence $M^\circ \hat{\otimes} M$ is also a weakly-permutation Mackey functor. Therefore M is an endo-permutation Mackey functor.

Example 2. An example of a capped endo-permutation Mackey functor is the Burnside functor kB^P . Given a subgroup $Q \leq P$, the evaluation $kB^P(Q)$ of the Burnside functor at Q is the k -linear extension $k \otimes B(Q)$ of the Burnside group $B(Q)$ of Q . Here the Burnside group $B(Q)$ is the Grothendieck group of the category of Q -sets with disjoint union as the addition. The action of the Mackey algebra is induced by the usual functors between categories of sets with a group action. For example, if $R \leq Q \leq P$, then r_R^Q act as the restriction map from the category of Q -sets to the category of R -sets.

Now, by [17, Proposition 6.2], we have $kB^P(Q) \cong k$ for each $Q \leq P$. Also, by [18, Corollary 8.9], the Burnside functor kB^P is indecomposable, isomorphic to the projective cover $\mathcal{P}_{P,1}$ of the simple functor $S_{P,1}$. Thus the Burnside functor kB^P is an indecomposable capped endo-permutation Mackey functor.

For the rest of the paper, we only consider capped endo-permutation Mackey functors and hence capped endo-permutation modules. Therefore, for the rest of the paper, we assume that all endo-permutation Mackey functors and all endo-permutation modules are capped. With this assumption, since the zero module is not a capped endo-permutation kP -module, for any finite p -group P , the following holds.

Corollary 7. Let M be a capped endo-permutation Mackey functor for P over k . Then $\overline{M}(Q)$ is non-zero for each subgroup $Q \leq P$.

Next we look at some properties of capped endo-permutation Mackey functors. Let M and N be capped endo-permutation Mackey functors for P and M' be a direct summand of M . Then, in general, M' is not a capped endo-permutation Mackey functor since we may have $\overline{M'}(Q) = 0$ for some subgroup $Q \leq P$. However, since any direct summand of an endo-permutation module is again endo-permutation, if $\overline{M'}(Q)$ is capped for each subgroup $Q \leq P$, then M' is a capped endo-permutation Mackey functor. For simplicity, a direct summand M' of M such that $\overline{M'}(Q) \neq 0$ for each subgroup $Q \leq P$ is called **primordial**. Thus, any primordial summand of a capped endo-permutation Mackey functor is a capped endo-permutation Mackey functor.

On the other hand, since the bar construction functor $M \mapsto \overline{M}$ is additive, the direct sum $M \oplus N$ is an endo-permutation Mackey functor provided that for each $Q \leq P$, the endo-permutation $kN_P(Q)/Q$ -modules $\overline{M}(Q)$ and $\overline{N}(Q)$ are compatible, that is, $\overline{M}(Q)^* \otimes \overline{N}(Q)$ is a permutation $kN_P(Q)/Q$ -module, [11, Proposition 2.3]. If this condition holds, then we say that M and N are **compatible** endo-permutation Mackey functors.

Moreover [1, Lemma 10.8.5] yields

$$\overline{M \hat{\otimes} N}(Q) \cong \overline{M}(Q) \otimes \overline{N}(Q)$$

for all Mackey functors M and N for P and for any subgroup Q of P . Here the tensor product on the right hand side is the usual tensor product over k of $kN_P(Q)/Q$ -modules. Thus, since the tensor product of endo-permutation modules is again an endo-permutation module, by [11, Proposition 2.2], the Mackey functor $M \hat{\otimes} N$ is also an endo-permutation Mackey functor.

Note that, in general, the class of capped endo-permutation Mackey functors is not closed under taking duals, since $\overline{M}^*(Q)$ may be zero even though $\overline{M}(Q)$ is not. However, taking twin-duals work. Indeed, if M is a Mackey functor, then by Equation (3), we have $\overline{M}^\circ(Q) = (\overline{M}(Q))^*$. Further if M is a capped endo-permutation Mackey functor, then the $kN_P(Q)/Q$ -module $\overline{M}(Q)$ is a capped endo-permutation module. Also, by [11, Proposition 2.2], the dual module $(\overline{M}(Q))^*$ is a capped endo-permutation module. Therefore, the twin-dual M° is a capped endo-permutation Mackey functor.

Finally, by Proposition 1 and by [5, Corollary 2.13], the class of endo-permutation Mackey functors is preserved under the functors of restriction, multiplicative deflation, multiplicative inflation and the transport of structure.

We summarize these properties with the following theorem.

Theorem 8. *The class of capped endo-permutation Mackey functors is closed under the operations of tensor products, taking primordial direct summands, taking twin-duals and the under the functors of restriction, multiplicative deflation, multiplicative inflation and the transport of structure introduced in the previous section. Moreover, direct sum of two compatible endo-permutation Mackey functors is also endo-permutation.*

Our next aim is to construct the group of capped endo-permutation Mackey functors. Following Dade, we want to consider an equivalence relation on the isomorphism classes of capped endo-permutation Mackey functors. To have a comparison, recall that two capped endo-permutation kP -modules m and n are called **equivalent**, written $m \sim n$ if they have isomorphic caps. To be more precise, note that the cap of a capped endo-permutation kP -module m is the, unique up to isomorphism, indecomposable summand of m with vertex P . Thus, we have $m \sim n$ if m and n have a common indecomposable direct summand with vertex P .

Now in the case of Mackey functors, by [16], any indecomposable Mackey functor M has a vertex. If M is an endo-permutation Mackey functor, then it has an indecomposable summand with vertex P . Indeed, since, by definition,

we have $\overline{M}(P) \neq 0$, there is an element $x \in M(P)$ such that the image \bar{x} of x in the quotient $\overline{M}(P)$ is not zero. Thus the subfunctor $\langle x \rangle$ of M generated by the element x has vertex P since, by its choice, x cannot be contained in $(\text{Ind}_Q^P \text{Res}_Q^P \langle x \rangle)(P)$. However this summand is not necessarily primordial. Therefore we cannot define an equivalence relation on these functors using this summand with vertex P . Instead, we consider the following relation.

Definition 9. *Let M and N be capped endo-permutation Mackey functors for P . We say that M and N are **equivalent** if for any subgroup $Q \leq P$, the capped endo-permutation $kN_P(Q)/Q$ -modules $\overline{M}(Q)$ and $\overline{N}(Q)$ are equivalent, with respect to Dade's equivalence of endo-permutation modules.*

It is straightforward to show that the above relation is an equivalence relation. Given a capped endo-permutation Mackey functor M , we denote by $[M]$ its equivalence class with respect to this equivalence relation. Moreover we denote the set of all equivalence classes of endo-permutation Mackey functors for P by

$$D_\mu(P).$$

This set becomes an abelian group under the operation

$$[M] + [N] = [M \widehat{\otimes} N] = [N \widehat{\otimes} M]$$

where M and N are endo-permutation Mackey functors for P . Here, we use [1, Proposition 1.9.1] for the commutativity of the tensor product. Note that the identity element with respect to this operation is the class $[kB^P]$ of the Burnside functor since

$$kB^P \widehat{\otimes} M \cong M \cong M \widehat{\otimes} kB^P$$

for any Mackey functor M for P , by Proposition 2.4.5 in [1]. Remark that the class $[kB^P]$ contains all endo-permutation Mackey functors N such that $[\overline{N}(Q)] = [k]$ in the Dade group $D(N_P(Q)/Q)$ of $N_P(Q)/Q$.

Regarding the inverses, as remarked earlier, the class of endo-permutation Mackey functors is not closed under taking duals but it is closed under taking twin-duals. We claim that the inverse of the class $[M]$ is the class containing M° . Indeed, if M is a capped endo-permutation Mackey functor for P , then M° is also a capped endo-permutation Mackey functor. Moreover by Equation (3), we have

$$\overline{(M^\circ \widehat{\otimes} M)}(Q) = \overline{M}(Q) \otimes \overline{M^\circ}(Q) = \overline{M}(Q) \otimes (\overline{M}(Q))^*.$$

Therefore, in $D_\mu(P)$, we have

$$[M^\circ \widehat{\otimes} M] = [kB^P].$$

Thus we have proved the following theorem.

Theorem 10. *Let P be a p -group and k be an algebraically closed field of characteristic p . Then the set $D_\mu(P)$ of the equivalence classes of endo-permutation Mackey functors for P over k is an abelian group where*

- (i) the group operation is given by the tensor product of Mackey functors,
- (ii) the identity element of the group is the class $[kB^P]$ of the Burnside functor, and
- (iii) given an endo-permutation Mackey functor M , the inverse $-[M]$ of the class $[M]$ is the class containing the twin-dual M° of M .

The set $D_\mu(P)$ together with this group operation is called the **Dade group** of Mackey functors for P , or, shortly, the **Mackey-Dade group** of P .

Remark 11. Let $R \leq P$. Then the restriction functor on Mackey functors induces a group homomorphism

$$\text{Res}_R^P : D_\mu(P) \rightarrow D_\mu(R).$$

Similarly, If $N \trianglelefteq P$ and $Q = P/N$, the multiplicative deflation and multiplicative inflation maps of the previous section induce group homomorphisms

$$\text{Jef}_Q^P : D_\mu(P) \rightarrow D_\mu(Q)$$

and

$$\text{Jnf}_Q^P : D_\mu(Q) \rightarrow D_\mu(P).$$

Also, if P and P' are isomorphic groups, then any isomorphism between them induces a transport of structure map between the corresponding Mackey-Dade groups. However, we do not know a way to construct a multiplicative induction functor for endo-permutation Mackey functors, and therefore, at this point, we cannot equip D_μ with a structure of a biset functor.

5. Relating the Mackey-Dade group to the Dade group

The Mackey-Dade group of P is closely related to the Dade groups of P and the Weyl groups of its subgroups, as the definition suggests. In this section, we clarify this relation by proving the following theorem.

Theorem 12. Let P and k be as above. Then there is an isomorphism of abelian groups

$$D_\mu(P) \cong \bigoplus_{Q \leq_P P} D(N_P(Q)/Q)$$

where the sum is over a set of representatives of conjugacy classes of subgroups of P . Moreover a canonical isomorphism is given by associating an endo-permutation Mackey functor M to the sequence $([\overline{M}(Q)])_{Q \leq_P P}$.

Proof Define

$$\overline{?} : D_\mu(P) \rightarrow \bigoplus_{Q \leq_P P} D(N_P(Q)/Q)$$

by $\overline{[M]} = ([\overline{M}(Q)])_{Q \leq_P P}$. Clearly, this is an abelian group homomorphism. To show that it is also bijective, we construct an inverse. Let $(m_Q)_{Q \leq_P P}$ be a

sequence in the direct sum. For each subgroup $Q \leq P$, denote by n_Q an endo-permutation $kN_P(Q)/Q$ -module representing the class m_Q . Also let C_{Q,n_Q} be the atomic conjugation functor concentrated on the conjugacy class of Q with $C_{Q,n_Q}(Q) = n_Q$. Finally, let

$$Y_{Q,n_Q} = \text{Ind}_P^\mu \text{Inf}_\gamma^\rho C_{Q,n_Q}.$$

With this notation, define the map

$$\varrho_P : \bigoplus_{Q \leq_P P} D(N_P(Q)/Q) \rightarrow D_\mu(P)$$

by

$$\varrho_P((m_Q)_{Q \leq_P P}) = \left[\bigoplus_{Q \leq_P P} Y_{Q,n_Q} \right].$$

For simplicity, we write Y for the Mackey functor $\bigoplus_{Q \leq_P P} Y_{Q,n_Q}$. Now we need to check that Y is a capped endo-permutation module, that is, we need to justify that the $kN_P(R)/R$ -module $\overline{Y}(R)$ is a capped endo-permutation module for each subgroup $R \leq P$. But by [9, Theorem 3.4.1 and Proposition 3.6], we have $\overline{Y_{Q,n_Q}}(R) = 0$ unless R is P -conjugate to Q and

$$\overline{Y_{R,n_R}}(R) = n_R$$

which is a capped endo-permutation $kN_P(R)/R$ -module by its choice. Thus Y is a capped endo-permutation Mackey functor.

On the other hand, to prove that the map ϱ_P is well-defined, let n'_Q be another representative of the class m_Q for each $Q \leq P$. Then since $\overline{Y_{Q,n_Q}}(Q) = n_Q$, for each $Q \leq P$, the equalities

$$\left[\bigoplus_{Q \leq_P P} Y_{Q,n_Q}(R) \right] = [n_R] = m_R = [n'_R] = \left[\bigoplus_{Q \leq_P P} Y_{Q,n'_Q}(R) \right].$$

hold in the Dade group of $N_P(R)/R$. Therefore, by definition, the equality

$$\left[\bigoplus_{Q \leq_P P} Y_{Q,n_Q} \right] = \left[\bigoplus_{Q \leq_P P} Y_{Q,n'_Q} \right]$$

holds in the Mackey-Dade group of P , that is, the map is well-defined. The above proof also shows that

$$\overline{\varrho_P((m_Q)_{Q \leq_P P})} = ([\overline{Y}(Q)])_{Q \leq_P P} = ([n_Q])_{Q \leq_P P} = (m_Q)_{Q \leq_P P}.$$

Hence, we have proved that the composition $\overline{\varrho_P}$ is the identity endomorphism of $\bigoplus_Q D(N_P(Q)/Q)$. On the other hand, if M is an endo-permutation Mackey functor, we have

$$\overline{\varrho_P([\overline{M}])}(R) = \overline{\bigoplus_{Q \leq_P P} Y_{Q,\overline{M}(Q)}(R)} = \overline{M}(R)$$

for each subgroup $R \leq P$. Therefore $\varrho_P(\bar{?}) = \text{id}_{D_\mu(P)}$, as required. \square

As a corollary to the above theorem, we determine the torsion-free rank of $D_\mu(P)$ as follows. For simplicity, we put $\mathbb{Q}D_\mu(P) = \mathbb{Q} \otimes_{\mathbb{Z}} D_\mu(P)$.

Corollary 13. *With the above notation, the \mathbb{Q} -dimension of $\mathbb{Q}D_\mu(P)$ is the number of conjugacy classes of non-cyclic subquotients of P , that is,*

$$\dim_{\mathbb{Q}} \mathbb{Q}D_\mu(P) = |\{(R, Q) | Q \trianglelefteq R \leq P : R/Q \text{ non-cyclic}\} / P|.$$

Proof By the above theorem, we have

$$\dim_{\mathbb{Q}} \mathbb{Q}D_\mu(P) = \sum_{Q \leq_P P} \dim_{\mathbb{Q}} \mathbb{Q}D(N_P(Q)/Q).$$

On the other hand, by [5, Theorem A], the torsion-free rank of $D(N_P(Q)/Q)$ is equal to the number $nc(N_P(Q)/Q)$ of conjugacy classes of non-cyclic subgroups of $N_P(Q)/Q$. Thus we get

$$\dim_{\mathbb{Q}} \mathbb{Q}D_\mu(P) = \sum_{Q \leq_P P} nc(N_P(Q)/Q).$$

But note that if R/N is a non-cyclic subquotient of P , then it is a subgroup of exactly one group of the form $N_P(Q)/Q$, namely of the one for which Q and N are conjugate. Therefore, as Q runs over a complete set of representatives of conjugacy classes of P and as R/Q runs over a complete set of representatives of conjugacy classes of non-cyclic subgroups of $N_P(Q)/Q$, the groups R/Q run over a complete set of representatives of conjugacy classes of non-cyclic subquotients of P . This completes the proof. \square

A similar result can be obtained for the torsion-part of the Mackey-Dade group. On the other hand, in the reverse direction, we can identify the Dade group $D(P)$ with a subgroup of the Mackey-Dade group $D_\mu(P)$ in a canonical way by identifying the Dade group in the above direct sum, as follows.

Corollary 14. *The above isomorphism ϱ_P induces a group monomorphism*

$$\varrho_P : D(P) \rightarrow D_\mu(P)$$

given by associating an endo-permutation kP -module m to the Mackey functor

$$\bigoplus_{Q \leq_P P} Y_{Q, m_Q}$$

where $m_Q = \text{Def}_{N_P(Q)/Q}^{N_P(Q)} \text{Res}_{N_P(Q)}^P m$ for each $Q \leq P$.

As another corollary, we have the following canonical representative of the equivalence class of an endo-permutation Mackey functor.

Corollary 15. *Let M be an endo-permutation Mackey functor for P . Then the class of M in $D_\mu(P)$ contains the Mackey functor*

$$YM := \bigoplus_{Q \leq_P P} Y_{Q, \overline{M}(Q)}.$$

The above corollary suggests another characterization of the equivalence classes of capped endo-permutation Mackey functors. Let M be a capped endo-permutation Mackey functor. For each $Q \leq P$, let $C(M, Q)$ denote the cap of the capped endo-permutation $kN_P(Q)/Q$ -module $\overline{M}(Q)$.

Definition 16. *Let M be a capped endo-permutation Mackey functor for P over k . We define the **cap** of M as the capped endo-permutation Mackey functor*

$$\text{cap}(M) = \bigoplus_{Q \leq_P P} Y_{Q, C(M, Q)}$$

for P over k .

Note that by [11, p. 470], the cap of a capped endo-permutation kP -module M is the unique, up to isomorphism, indecomposable summand of M with vertex P . As we have explained above, a capped endo-permutation Mackey functor M for P may not have an indecomposable summand with vertex P which is a capped endo-permutation Mackey functor. With the above definition, the cap of M is the unique, up to isomorphism, primordial summand $\text{cap}(M)$ of M such that the vertex of the $kN_P(Q)/Q$ -module $\overline{\text{cap}(M)}(Q)$ is $N_P(Q)/Q$ for each subgroup $Q \leq P$.

Now the following corollary is immediate by the definition of Dade's equivalence of endo-permutation modules and our definition of equivalence of endo-permutation Mackey functors.

Corollary 17. *Let M and N be endo-permutation Mackey functors. Then M is equivalent to N if and only if there is an isomorphism*

$$\text{cap}(M) \cong \text{cap}(N)$$

of Mackey functors for P .

6. Ring of subquotients and the Mackey-Dade group

In this section, we show that the Mackey-Dade group $\mathbb{Q}D_\mu(P)$ tensored with \mathbb{Q} over \mathbb{Z} can be realized as the kernel of the linearization map defined by the author in [10]. First we recall the definitions that we need.

Let P be a finite p -group. We write $\mathcal{SQ}(P)$ for the set of all subquotients of P , that is,

$$\mathcal{SQ}(P) = \{(Q, N) | N \trianglelefteq Q \leq P\}$$

and write $\mathcal{SQ}_P(P)$ for the set of P -orbits of the set \mathcal{SQ} . We sometimes write the pair (Q, N) as Q/N , and $(Q, N) \in \mathcal{SQ}(P)$ as $Q/N \preceq P$. Also, the P -orbit containing (Q, N) is denoted by $[Q, N]_P$ and we write $Q/N \preceq_P P$ instead of $[Q, N]_P \in \mathcal{SQ}_P(P)$.

Now we denote by $\Lambda(P)$ the free abelian group on the set $\mathcal{SQ}_P(P)$, that is,

$$\Lambda(P) = \bigoplus_{Q/N \preceq_P P} \mathbb{Z}[Q, N]_P.$$

This group becomes a unital commutative associative ring with the product given by the linear extension of the product defined on the basis elements by

$$[R, N]_P \cdot [S, M]_P = \sum_{\substack{x \in R \setminus P/S \\ {}^xMN \leq R \cap {}^xS}} [R \cap {}^xS, {}^xMN].$$

The unity of this ring is $[P/1]_P$. We call $\Lambda(P)$ the **ring of subquotients** of P . Note that the Burside ring is a subring of $\Lambda(P)$ via the inclusion

$$[P/R] \mapsto [R, 1]_P.$$

We refer to [10] for further details on the structure of $\Lambda(P)$.

The reason for introducing the above product on subquotients is to define a linearization map for Mackey functors. By [18], the Mackey algebra for P over a field \mathbb{K} of characteristic zero is semi-simple. We denote by $\mathcal{R}_{\mu_{\mathbb{K}}}(P)$ the Grothendieck ring of the category of Mackey functors for P over \mathbb{K} . Here the addition and the multiplication operations are given, respectively, by the direct sum and the tensor product of Mackey functors. The set of isomorphism classes of simple Mackey functors forms a basis for the ring $\mathcal{R}_{\mu_{\mathbb{K}}}(P)$ and the class of the Burnside ring functor $\mathbb{K}B^P$ is the unit element. Therefore, by the classification of simple Mackey functors given in [19], there is an isomorphism

$$\mathcal{R}_{\mu_{\mathbb{K}}}(P) \cong \bigoplus_{\substack{(H,V) \\ H \leq P, V \in \text{Irr}(\mathbb{K}N_P(H)/H)}} \mathbb{Z}[S_{H,V}^P]$$

of abelian groups where $[S_{H,V}^P]$ denotes the isomorphism class of the simple Mackey functor $S_{H,V}^P$ for P with minimal subgroup H and $S_{H,V}^P(H) = V$.

Now the linearization map

$$\text{Lin}_P^\mu : \Lambda(P) \rightarrow \mathcal{R}_{\mu_{\mathbb{K}}}(P)$$

is defined as the linear extension of the map associating the basis element $[Q, N]_P$ to the class of the Mackey functor $\text{Ind}_Q^P \text{Inf}_{Q/N}^Q \mathbb{K}B^{Q/N}$. Here the inflation and induction maps are the usual ones as defined in [18], or as in [1].

By [10, Theorem 5.3], this linearization map is a ring homomorphism. Also by [10, Theorem 5.5], the extension

$$\mathbb{Q}\text{Lin}_P^\mu : \mathbb{Q}\Lambda(P) \rightarrow \mathbb{Q}\mathcal{R}_{\mu_{\mathbb{Q}}}(P)$$

is surjective. Hence, we have an analogy with the well-known linearization map

$$\text{Lin}_P : B(P) \rightarrow \mathcal{R}_{\mathbb{K}}(P)$$

where $\mathcal{R}_{\mathbb{K}}(P)$ is the \mathbb{K} -character ring of P and the map is given by associating the basis element $[P/Q]$ of the Burnside ring $B^P(P)$ to the permutation character $\text{Ind}_Q^P \mathbb{K} = \text{Ind}_Q^P \text{Inf}_{Q/Q}^Q \mathbb{K}$ where \mathbb{K} is the trivial $\mathbb{K}G$ -module for any finite group G .

To push the analogy further, recall that by [5, Theorem D], the kernel of $\mathbb{Q}\text{Lin}_P$ is the torsion-free group $\mathbb{Q}D(P) = \mathbb{K} \otimes_{\mathbb{Z}} D(P)$. Next, we show that this is also the case for Mackey functors. However, as we shall see in the next section, this identification is functorial only in a restricted sense.

Henceforth, let P be a finite p -group and let $s(P)$ denote a set of representatives of the conjugacy classes of subgroups of P . To prove the analogue of [5, Proposition 5.1] in terms of Mackey functors, we need the following result.

Theorem 18. [18, Proposition 6.1] *Let P be a p -group and let $s(P)$ be as above. Then the bar construction induces an isomorphism of \mathbb{Q} -vector spaces*

$$\bar{\cdot} : \mathbb{Q}\mathcal{R}_{\mu_{\mathbb{K}}}(P) \rightarrow \bigoplus_{R \in s(P)} \mathbb{Q}\mathcal{R}_{\mathbb{K}}(N_P(R)/R)$$

which maps a Mackey functor M to the sequence $(\overline{M}(R))_{R \in s(P)}$. The inverse of $\bar{\cdot}$ maps a sequence $(m_R)_{R \in s(P)}$ to the Mackey functor $\bigoplus_{R \in s(P)} Y_{R, m_R}$.

A similar result holds for the space $\mathbb{Q}\Lambda(P)$ of subquotients. Next we describe this result.

Theorem 19. *Assume the above notation. Then the linear map*

$$\alpha_P : \mathbb{Q}\Lambda(P) \rightarrow \bigoplus_{R \in s(P)} \mathbb{Q}B(N_P(R)/R)$$

extending the function given by mapping a basis element $[Q, N]_P$ of $\mathbb{Q}\Lambda(P)$ to the sequence $(a_R(Q, N))_{R \in s(P)}$ where

$$a_R(Q, N) = \sum_{\substack{g \in Q \backslash P / N_P(R) : \\ N \leq^g R \leq Q}} [N_P(R) / N_{g^{-1}Q}(R)]$$

is an isomorphism of \mathbb{Q} -vector spaces.

Proof First, we check that the map α_P is well-defined. Let $(T, S) = {}^x(Q, N)$ for some $x \in P$. Then

$$\begin{aligned} a_R(T, S) = a_R({}^xQ, {}^xN) &= \sum_{\substack{h \in {}^xQ \backslash P / N_P(R) : \\ {}^xN \leq^h R \leq {}^xQ}} [N_P(R) / N_{h^{-1}{}^xQ}(R)] \\ &= \sum_{\substack{x^{-1}h \in Q \backslash P / N_P(R) : \\ N \leq^{x^{-1}h} R \leq Q}} [N_P(R) / N_{h^{-1}{}^xQ}(R)] \\ &= \sum_{\substack{g \in Q \backslash P / N_P(R) : \\ N \leq^g R \leq Q}} [N_P(R) / N_{g^{-1}Q}(R)] \\ &= a_R(Q, N) \end{aligned}$$

as required. Thus the map α_P is well-defined.

By definition α_P is \mathbb{Q} -linear. To prove that it is bijective, we first show that the domain and the range of α_P have the same \mathbb{Q} -dimension, and then show that the map α_P is injective. Recall that the set $\mathcal{SQ}_P(P)$ of all P -conjugacy classes of all pairs (Q, N) with $N \trianglelefteq Q \leq P$ is a basis for the vector space $\mathbb{Q}\Lambda(P)$. On the other hand, a basis for the vector space

$$\mathbb{Q}B^\oplus(P) := \bigoplus_{R \in s(P)} \mathbb{Q}B(N_P(R)/R)$$

is the set

$$\mathcal{SS}_P(P) := \bigcup_{R \in s(P)} \mathcal{S}(N_P(R)/R)$$

where $\mathcal{S}_G(G)$ denotes the set of conjugacy classes of subgroups of G for any finite group G . For simplicity, we write $[T]_R$ for the $N_P(R)/R$ -conjugacy class of T/R . Therefore our claim is equivalent to prove that $|\mathcal{SQ}_P(P)| = |\mathcal{SS}_P(P)|$. We define two functions

$$\begin{aligned} \tilde{f}: \mathcal{SS}_P(P) &\rightarrow \mathcal{SQ}_P(P) \\ [T]_R &\mapsto [T, R]_P. \end{aligned}$$

and

$$\begin{aligned} f: \mathcal{SQ}_P(P) &\rightarrow \mathcal{SS}_P(P) \\ [Q, N]_P &\mapsto [\{Q\}]_{\{N\}} \end{aligned}$$

where $\{N\}$ denotes the representative of the P -conjugacy class of N in the set $s(P)$ and putting $\{N\} = {}^xN$, we define $\{Q\}$ as the $N_P(\{N\})/\{N\}$ -conjugacy class of ${}^xQ/{}^xN$. We have to show that f and \tilde{f} are well-defined mutually inverse functions.

The function \tilde{f} is well-defined: if $[T]_R = [T']_{R'}$, then $R = R'$, and if $T' = {}^xT$ for some $x \in N_P(R)$, we get

$$\tilde{f}([T']_R) = [T', R]_P = [{}^xT, R]_P = [{}^xT, {}^xR]_P = [T, R]_P = \tilde{f}([T]_R).$$

To see that the function f is well-defined, let $(L, M) = {}^g(Q, N)$ for some $g \in P$. In particular, the subgroups M and N are P -conjugate and hence $\{M\} = \{N\}$. Let ${}^xN = \{M\} = {}^yM$. Then

$$[{}^{y^{-1}x}L] = [{}^{y^{-1}xg}Q] = [Q]$$

as $N_P(M)$ -conjugacy classes since by the above choices, we have $y^{-1}xg \in N_P(M)$. Thus, $[{}^xL] = [{}^xQ]$ as $N_P(\{M\})$ -conjugacy classes and thus the function f is well-defined.

Next, we show that the functions f and \tilde{f} are inverse functions. We have

$$(\tilde{f} \circ f)([Q, N]_P) = \tilde{f}([\{Q\}]_{\{N\}}) = [\{Q\}, \{N\}]_P.$$

Since $(\{Q\}, \{N\})$ is P -conjugate to (Q, N) , by definition, we get $\tilde{f} \circ f = \text{id}_{\mathcal{SQ}_P(P)}$. Similarly,

$$(f \circ \tilde{f})([T]_R) = f([T, R]_P) = [\{T\}]_{\{R\}}.$$

Since $R \in s(P)$, we have $\{R\} = R$, and hence $\{T\}$ is $N_P(R)$ -conjugate to T . Therefore $[\{T\}]_{\{R\}} = [T]_R$ and hence $f \circ \tilde{f} = \text{id}_{\mathcal{SS}_P(P)}$.

Finally, to complete the proof, we show that the map α_P is injective. Let

$$u = \sum_{Q/N \preceq_P P} u_{Q,N} [Q, N]_P$$

such that not all $u_{Q,N}$ are zero but $\alpha_P(u) = 0$. Let R be a subgroup of P of minimal order such that $u_{Q,R} \neq 0$ for some $Q \leq P$ and, without loss, suppose that $R \in s(P)$. Then, since $\alpha_P(u) = 0$,

$$\begin{aligned} 0 = a_R(u) &= \sum_{Q/N \preceq_P P} u_{Q,N} a_R(Q, N) \\ &= \sum_{Q/N \preceq_P P} u_{Q,N} \sum_{\substack{g \in Q \backslash P / N_P(R) \\ N \leq {}^g R \leq Q}} [N_P(R) / N_{g^{-1}Q}(R)]. \end{aligned}$$

By the choice of R , the only non-zero contribution to the sum comes from the pairs where $N = {}^g R$ for some $g \in P$. Since N and R are both taken up to P -conjugacy, there is only one possible choice for N and since $a_R(Q, N)$ is independent of the choice of the representative (Q, N) of the class $[Q, N]_P$, we may let $N = R$. In this case, the condition $N = {}^g R$ implies that $g \in N_P(R)$, and hence there is only the trivial $(Q, N_P(R))$ -double coset to consider in the second sum of the last sum above. Moreover, the indices of the former sum run through the conjugacy classes subgroups of $N_P(R)$ containing R . Hence, we obtain

$$0 = a_R(u) = \sum_{R \leq Q \leq_{N_P(R)} N_P(R)} u_{Q,R} [N_P(R) / Q].$$

Now, since the set $\{[N_P(R) / Q] \mid R \leq Q \leq_{N_P(R)} N_P(R)\}$ is a basis for the \mathbb{Q} -vector space $\mathbb{Q}B(N_P(R) / R)$, the above equality implies that $u_{Q,R} = 0$ for each Q . But this contradicts to the choice of R . Hence there is no such u and α_P is injective. \square

The above isomorphism and the isomorphism of Theorem 18 are related by the following commutative diagram.

Corollary 20. *Assume the above notation, and let Lin_P^\oplus denote the sum of the*

linearization maps $\text{Lin}_{N_P(R)/R}$ over $R \in s(P)$. Then the diagram

$$\begin{array}{ccc}
\mathbb{Q}\Lambda(P) & \xrightarrow{\text{Lin}_P^\mu} & \mathbb{Q}\mathcal{R}_{\mu_{\mathbb{Q}}}(P) \\
\downarrow \alpha_P & & \downarrow \bar{?} \\
\bigoplus_{R \in s(P)} \mathbb{Q}B(N_P(R)/R) & \xrightarrow{\text{Lin}_P^\oplus} & \bigoplus_{R \in s(P)} \mathbb{Q}\mathcal{R}_{\mathbb{K}}(N_P(R)/R)
\end{array}$$

commutes.

Proof Since all of the maps shown in the diagram are linear, it is sufficient to prove the claim for a basis element. Hence let R/N be a subquotient of P . We need to show that

$$\overline{\text{Lin}_P^\mu([R/N]_P)} = \text{Lin}_P^\oplus(\alpha_P([R/N]_P)).$$

Note that, by definition, we have

$$\text{Lin}_P^\mu([R/N]_P) = \text{Ind}_R^P \text{Inf}_{R/N}^R B^{R/N}.$$

However by [10, Section 6], we have

$$\text{Lin}_P^\mu([R/N]_P) = \bigoplus_{N \leq K \leq_R R} Y_{K, \text{Ind}_{W_R(K)}^{W_P(K)}} 1.$$

Here we write $W_R(K) = N_R(K)/K$ to simplify the notation. Also, since for any $W_P(K)$ -module m , we have $\overline{Y_{K,m}}(Q) = 0$ unless $K =_P Q$ and $\overline{Y_{K,m}}(K) = m$, it follows that the Q -th component of $\overline{\text{Lin}_P^\mu([R/N]_P)}$ is

$$(\overline{\text{Lin}_P^\mu([R/N]_P)})_Q \cong \bigoplus_{\substack{N \leq K \leq_R R \\ K =_P Q}} \text{Ind}_{W_{g^{-1}R}^{W_P(Q)}} 1 \quad (4)$$

where, for each P -conjugate K of Q , the element g is chosen so that $K = {}^g Q$. On the other hand, by the definition of α_P , the Q -th component of $\alpha_P([R/N]_P)$ is

$$a_Q([R/N]_P) = \sum_{g \in R \setminus P/N_P(Q) : N \leq {}^g Q \leq R} [N_P(Q)/N_{g^{-1}R}(Q)]$$

Thus the Q -th component of $\text{Lin}_P^\oplus(\alpha_P([R/N]_P))$ is

$$(\text{Lin}_P^\oplus(\alpha_P([R/N]_P)))_Q \cong \bigoplus_{\substack{g \in R \setminus P/N_P(Q) \\ N \leq {}^g Q \leq R}} \text{Ind}_{N_{g^{-1}R}^{N_P(Q)}} 1. \quad (5)$$

Finally, since the index set of Equation (4) and Equation (5) are the same, equality holds and the diagram commutes. \square

As another corollary of the above theorem, we get an isomorphism

$$\ker(\text{Lin}_P^\mu) \cong \ker(\text{Lin}_P^\oplus).$$

Now by [5, Theorem D], for any finite p -group P , we have

$$\mathbb{Q}D(P) \cong \ker(\text{Lin}_P : \mathbb{Q}B(P) \rightarrow \mathbb{Q}\mathcal{R}_\mathbb{Q}(P)).$$

Therefore we have

$$\ker(\text{Lin}_P^\oplus) \cong \bigoplus_{R \in s(P)} \mathbb{Q}D(N_P(R)/R).$$

Thus, by Theorem 12, we get an isomorphism

$$\ker(\text{Lin}_P^\mu) \cong \mathbb{Q}D_\mu(P)$$

together with the map

$$\iota_P^\mu := \alpha_P^{-1} \circ \iota_P^\oplus \circ \bar{?} : \mathbb{Q}D_\mu(P) \rightarrow \mathbb{Q}\Lambda(P)$$

where we write $\iota_P : \mathbb{Q}D(P) \rightarrow \mathbb{Q}B(P)$ for the map α of [5, Theorem D] and ι_P^\oplus for the corresponding direct sum of $\iota_{N_P(R)/R}$'s. Now since α_P and $\bar{?}$ are isomorphisms and ι_P^\oplus is injective, the map ι_P^μ is also injective. Moreover, by the commutativity of the above diagram, for any endo-permutation Mackey functor M , we have

$$(\text{Lin}_P^\mu \circ \iota_P^\mu)(M) = \text{Lin}_P^\mu((\alpha_P^{-1} \circ \iota_P^\oplus)(\bar{M})) = (\varrho_P \circ \text{Lin}_P^\oplus \circ \iota_P^\oplus)(\bar{M}) = 0$$

Therefore, we obtain the following result.

Theorem 21. *Let P be a p -group. The sequence*

$$0 \longrightarrow \mathbb{Q}D_\mu(P) \xrightarrow{\iota_P^\mu} \mathbb{Q}\Lambda(P) \xrightarrow{\text{Lin}_P^\mu} \mathbb{Q}\mathcal{R}_{\mu_\mathbb{Q}}(P) \longrightarrow 0$$

of \mathbb{Q} -vector spaces is exact.

We finish by a result to show that the above analogy breaks for the following next step. By [15, Statement 2.1.2], there is an isomorphism

$$T(P) \cong \bigcap_{1 < Q \leq P} \text{Ker}(\text{Def}_{N_P(Q)/Q}^{N_P(Q)} \text{Res}_{N_P(Q)}^P : D(P) \rightarrow D(N_P(Q)/Q))$$

where we write $T(P)$ for the group of endo-trivial kP -modules. Our next result shows that the above analogy between endo-permutation Mackey functors and endo-permutation modules does not hold in this direction. Indeed, since we have defined restriction and deflation maps for the Mackey-Dade groups, the version of the right-hand side can be evaluated for the Mackey-Dade groups. However,

as we shall see, it is not isomorphic to the direct sum of the groups of endotrivial $kN_P(Q)/Q$ -modules as Q runs over all subgroup of P up to conjugation. In a future paper, we are aiming to study endotrivial Mackey functors and their relations to endo-permutation Mackey functors.

To describe the intersection of kernels of deflation-restriction maps for the Mackey-Dade group, we need the following definition of the endo-permutation module $\Delta(P)$ from [3]. Let $M(P)$ denote the disjoint union of the P -sets P/Q as Q runs over all maximal subgroups of the finite p -group P and let

$$\Delta(P) = \Omega_{M(P)}$$

denote the corresponding relative syzygy, as defined by Bouc in [3, Notation 6.2.1]. Then by [3, Remark 6.2.2], the class of $\text{Res}_R^P \Delta(P)$ in the Dade group $\mathbb{Q}D(R)$ is zero when $R \neq P$ and moreover $\Delta(P)$ is $\text{Out}(P)$ -invariant. Finally, by [3, Remark 7.4.11], the elements $\text{Ten}_R^P \Delta(R)$ form a basis of $\mathbb{Q}D(P)$, as R runs over all non-cyclic subgroups of P .

Proposition 22. *Let P be a non-cyclic p -group. Then*

$$\underline{\mathbb{Q}D}_\mu(P) := \bigcap_{R/N \leq P} \ker(\text{Jef}_{R/N}^R \text{Res}_R^P : \mathbb{Q}D_\mu(P) \rightarrow \mathbb{Q}D_\mu(R/N))$$

is equal to the $\mathbb{Q}\text{Out}(P)$ -submodule of $\mathbb{Q}D_\mu(P)$ generated by $\varrho_P(\Delta(P))$.

Proof Let M be an endo-permutation Mackey functor for P and let $1 \neq Q \leq P$. Then by Theorem 1, we have

$$\overline{\text{Jef}_{N_P(Q)/Q}^{N_P(Q)} \text{Res}_{N_P(Q)}^P M(Q/Q)} \cong \overline{M}(Q)$$

as $kN_P(Q)/Q$ -modules. This implies, in particular, that if M is in $\underline{D}_\mu(P)$, then $\overline{M}(Q)$ is zero in $\mathbb{Q}D(N_P(Q)/Q)$. In other words, $[M] \in \underline{\mathbb{Q}D}_\mu(P)$ only if $[\overline{M}(Q)] = 0$ for any $1 \neq Q \leq P$.

Regarding the endo-permutation kP -module $M(1)$, we have

$$(\text{Res}_R^P M)(1) = \text{Res}_R^P(M(1)).$$

Thus $[M] \in \underline{\mathbb{Q}D}_\mu(P)$ implies that $\text{Res}_R^P[M(1)] = 0$ in $\mathbb{Q}D(R)$ for any $R < P$. Since the proper subgroups R of P are the only subquotients of P for which the action of an (R, P) -biset contribute to $M(1)$, we have that $[M] \in \underline{\mathbb{Q}D}_\mu(P)$ if and only if

- (i) $[\overline{M}(Q)] = 0$ in $\mathbb{Q}D(N_P(Q)/Q)$ for each non-trivial subgroup $1 \neq Q \leq P$ and
- (ii) $\text{Res}_R^P[\overline{M}(1)] = 0$ in $\mathbb{Q}D(R)$ for each proper subgroup $R < P$.

Therefore, to determine the subgroup $\underline{\mathbb{Q}D}_\mu(P)$, we only need to calculate the subgroup

$$\underline{\mathbb{Q}D}(P) = \bigcap_{R < P} \ker(\text{Res}_R^P : \mathbb{Q}D(P) \rightarrow \mathbb{Q}D(R)).$$

We claim that this subgroup is generated by the endo-permutation kP -module $\Delta(P)$. Indeed, let $x \in \underline{\mathbb{Q}D}(P)$. Since the elements $\text{Ten}_R^P \Delta(R)$ form a basis of $\mathbb{Q}D(P)$, we can write

$$x = \sum_{R \leq_P P} a_R \text{Ten}_R^P \Delta(R)$$

for some $a_R \in \mathbb{Q}$. We want to show that all the coefficients a_Q are zero except possibly for a_P . To prove this claim, note that, for a fixed subgroup $R \leq P$, and a subgroup $Q < P$, the Mackey formula implies

$$\begin{aligned} \text{Res}_Q^P \text{Ten}_R^P \Delta(R) &= \sum_{x \in Q \backslash P/R} \text{Ten}_{Q \cap^x R}^Q \text{Res}_{Q \cap^x R}^{xR} \Delta({}^x R) \\ &= \sum_{\substack{x \in Q \backslash P/R \\ {}^x R \leq Q}} \text{Ten}_{{}^x R}^Q \Delta({}^x R). \end{aligned}$$

Here we also use that the restriction of $\Delta(R)$ to any proper subgroup of R is zero. In particular, if the order of Q is less than or equal to the order of R (and if Q is not P -conjugate to R , when they have equal orders), then the condition ${}^x R \leq Q$ is not satisfied for any $x \in P$ and we get

$$\text{Res}_Q^P \text{Ten}_R^P \Delta(R) = 0.$$

Now, let $Q \leq P$ be a subgroup of minimal order such that the coefficient $a_Q \neq 0$ and suppose that $Q \neq P$. Then by the above calculation and by the choice of x , we have

$$\begin{aligned} 0 = \text{Res}_Q^P x &= \sum_{R \leq_P P} a_R \text{Res}_Q^P \text{Ten}_R^P \Delta(R) \\ &= a_Q \text{Res}_Q^P \text{Ten}_Q^P \Delta(Q) \\ &= a_Q |N_P(Q) : Q| \Delta(Q). \end{aligned}$$

Therefore, we get that $a_Q = 0$, a contradiction. Hence we have $Q = P$, and $x = a_P \Delta(P)$ for some $a_P \in \mathbb{Q}$.

Finally, with the above identification of $\underline{\mathbb{Q}D}_\mu(P)$, we get that

$$\underline{\mathbb{Q}D}_\mu(P) \cong \langle \varrho_P(\Delta(P)) \rangle$$

where ϱ_P is as defined in Section 5. This isomorphism is an isomorphism of $\mathbb{Q}\text{Out}(P)$ -modules since $\Delta(P)$ is $\text{Out}(P)$ -invariant. \square

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References

- [1] S. Bouc, ‘Green functors and G -sets’, Lecture Notes in Mathematics, vol. 1671 Springer-Verlag, Berlin, (1997).
- [2] S. Bouc, ‘Résolutions de foncteurs de Mackey’, in: Group representations: Cohomology, Group Actions and Topology, AMS Proc. Symp. Pure Math. 63 (1998).
- [3] S. Bouc, ‘Tensor induction of relative syzygies’, *J. Reine Angew. Math.* 523 (2000) 113-171.
- [4] S. Bouc, ‘Non-additive exact functors and tensor induction for Mackey functors’, *AMS Memoirs* (2000) 144-683 .
- [5] S. Bouc, J. Thévenaz, ‘The group of endo-permutation modules’, *Invent. Math.* 139 (2000) 275-349.
- [6] S. Bouc, ‘The Dade group of a p -group’, *Invent. Math.* 164 (2006), 189-231.
- [7] S. Bouc, ‘Biset functors for finite groups’, Lecture Notes in Mathematics, vol. 1990 Springer-Verlag, Berlin (2010).
- [8] S. Bouc, ‘The slice Burnside ring and the section Burnside ring of a finite group’, *Compositio Math.* 148 (2012) 868-906.
- [9] O. Coşkun, ‘Mackey functors, induction from restriction functors and coinduction from transfer functors’, *J. Algebra* 315 (2007), 224-248.
- [10] O. Coşkun, ‘Ring of subquotients of a finite group I: Linearization’, *J. Algebra* 322 (2009) 2773 - 2792.
- [11] E. Dade, ‘Endo-permutation modules over p -groups I, II’, *Annals of Mathematics* 107 (1978) 459-494, 317-346.
- [12] A. Dress, ‘Contributions to the theory of induced representations’, Lecture Notes in Mathematics, vol. 342, pp. 183-240, Springer, (1973).
- [13] C. Lassueur, ‘The Dade group of a finite group’, *J. Pure and Appl. Algebra* 217 (2013) 97-113.
- [14] M. Linckelmann, N. Mazza, ‘The Dade group of a fusion system’, *J. Group Theory* 12 (2009) 55-74.
- [15] L. Puig, ‘Affirmative answer to a question of Feit’, *J. Algebra* 131 (1990), 513-526.
- [16] H. Sasaki, ‘Green correspondence and transfer theorems of Wielandt type for G -functors’, *J. Algebra* 79 (1982), 98-120.
- [17] J. Thévenaz, ‘Some remarks on G -functors and the Brauer morphisms’, *J. Reine Angew. Math.* 384 (1988), 24-56.

- [18] J. Thévenaz, P. Webb, 'The structure of Mackey functors', *Trans. Amer. Math. Soc.* 347 (1995), 1865-1961.
- [19] J. Thévenaz, P. Webb, 'Simple Mackey functors', Proceedings of the Second International Group Theory Conference (Bressanone, 1989). *Rend. Circ. Mat. Palermo (2)* Suppl. No. 23 (1990) 299-319.
- [20] J.M. Urfer, 'Endo-p-permutation modules', *J. Algebra* 316 (2007) 206-223.
- [21] P. Webb, 'Stratifications and Mackey functors II: globally defined Mackey functors', *J. K-Theory* 6 (2010) 99-170.